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Continuity of Lifting Surface Theory in Subsonic and Supersonic Flow

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The continuity of linear theory for lifting surfaces in subsonic and supersonic flow is discussed. The unsteady pressure kernel functions of the singular integral equation are summarized for all Mach numbers in the Laplace transform domain. The sonic kernel in the steady flow is obtained as a finite limit from both the subsonic and supersonic sides when the Mach number tends to unity. Numerical examples are given using the doublet-point method with three different kernel functions, showing continuous results through the unit Mach number. The results also show how a rectangular wing with various Mach numbers approaches to the two-dimensional airfoil as the aspect ratio increases.

Introduction

THE basic equation for lifting surfaces in unsteady potential flow was formulated by Küssner[1] in 1940. The equation has an integral form, and its kernel function includes differential operators that have to be inverted before reaching the solution for lift distributions. The equation relates directly the lift distribution and the disturbance velocity normal to the thin surfaces and is based on Prandtl's acceleration potential.[2]

Watkins et al.[3] intensively investigated the kernel function of subsonic flow by deriving relevant forms for numerical calculation. Many efforts were devoted to obtaining solutions,[4,5] including the case of the steady flow that corresponds to a zero reduced frequency. Attempts have also been made for supersonic flow to obtain load distributions on the thin wings by solving the singular integral equation with the pressure kernel function.[6,7]

As for the formulation of the kernel function in the general configuration including noncoplanar surfaces, Landahl[8] derived a compact form of the unsteady kernel for subsonic flow. Harder and Rodden[9] also formulated kernel functions for nonplanar surfaces in supersonic flow. These pressure kernel functions led to the advent of the doublet-point method (DPM), which unified subsonic[10] and supersonic[11] flow, using discrete elements in the numerical calculation.

Although the lifting surface theory should be continuous through all Mach numbers, not much effort has been devoted to clarifying the flow characteristics with a sonic Mach number, i.e., $M = 1$. This may be attributed to two facts. One is the completely singular behavior of two-dimensional flow at a unit Mach number, which is related to the Prandtl–Glauert factor[12] of the subsonic similarity rule collapsing at the sonic flow. The other issue is an unrealistic assumption of the linear lifting surfaces in sonic flow, which cannot take into consideration the wing thickness to render the intrinsic nonlinear effects on a real wing. In spite of these issues, it is worthwhile to investigate the continuity of the integral equations and their solutions through the unit Mach number to comprehend the mathematical behavior of the linear theory as a limiting case of the actual flow.

The purpose of the present paper is to summarize the lifting surface theory for all Mach numbers and to clarify the finite continuity of the noncoplanar kernel functions for steady flow with sonic Mach numbers, which can be realized as a limiting case for both subsonic and supersonic flow. The continuity of the numerical solutions is also examined for finite thin wings in sonic flow by using the DPM.

General Equations of the Lifting Surface Theory

The unsteady lifting surface theory for subsonic and supersonic flow can be summarized for the noncoplanar configuration as follows:

The basic singular integral equation in the Laplace domain that relates the normal wash W_n perpendicular to the uniform flow at a certain point (x, s) in three-dimensional space to the lift distribution $L(\xi, \sigma)$ on the lifting surfaces can be written as

$$W_n(x, s) = \frac{1}{8\pi}\iint L(\xi, \sigma)K(p, M)\,\mathrm{d}\xi\,\mathrm{d}\sigma \tag{1}$$

where K represents the kernel function dependent on the nondimensional Laplace variable p and Mach number M. In the present paper, all of the parameters are treated as nondimensional quantities unless otherwise mentioned. The lift distribution is assumed to be normalized by the dynamic pressure of the uniform flow. The variable p is a complex value in general and can be written with the so-called reduced frequency k as

$$p = \bar{s}b/U = h + ik \tag{2}$$

where the symbols $\bar{s}$, h, and k denote the dimensional variable of the Laplace transform, its nondimensional real part, and the imaginary part, respectively. The dimensional quantities b and U are a representative length and a uniform flow speed. The kernel function K has a general form as

$$K = e^{-px_0}(T_1K_1 - T_2K_2) \tag{3}$$

In this expression, T_1 and T_2 are the geometric factors for noncoplanar configuration and can be given by

$$T_1 = \cos[\gamma(y) - \gamma(\eta)] \tag{4}$$

and

$$T_2 = [z_0\cos\gamma(y) - y_0\sin\gamma(y)][z_0\cos\gamma(\eta) - y_0\sin\gamma(\eta)] \tag{5}$$

The geometry of the lifting surface in the Cartesian coordinates is shown in Fig. 1, and the variables are defined as

$$x_0 = x - \xi, \qquad y_0 = y - \eta, \qquad z_0 = z - \zeta \tag{6}$$

and

$$R = \sqrt{x_0^2 + (1 - M^2)r^2}, \quad \text{where} \quad r = \sqrt{y_0^2 + z_0^2} \tag{7}$$

Further, if the parameters in subsonic flow

$$X = \frac{x_0 - MR}{1 - M^2} \tag{8}$$

$$d = \sqrt{X^2 + r^2} = R - MX = M^{-1}(x_0 - X) \tag{9}$$

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Table 1 Unsteady kernel functions

Kernel functions	$M<1$	$M>1$
K_1	$(M/Rd)\,e^{pX}+B_1(p,r,X)$	$(M/Rd_1)\,e^{-pX_1}+(M/Rd_2)\,e^{-pX_2}$ $+B_1(-p,r,X_2)-B_1(-p,r,X_1)$
K_2	$\dfrac{M}{R^3d^3}\,e^{pX}(pMRd^2+3Rd-M^2r^2)$ $+3B_2(p,r,X)$	$\dfrac{M}{R^3d_1^3}\,e^{-pX_1}\left(pMRd_1^2+3Rd_1-M^2r^2\right)$ $-\dfrac{M}{R^3d_2^3}\,e^{-pX_2}\left(pMRd_2^2+3Rd_2+M^2r^2\right)$ $+3B_2(-p,r,X_2)-3B_2(-p,r,X_1)$

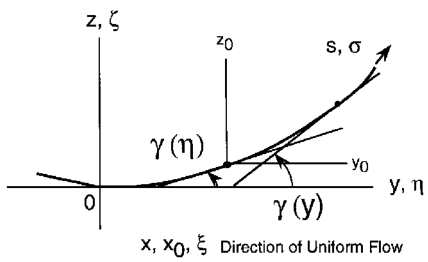

Fig. 1 Geometry of lifting surfaces.

or in supersonic flow

$$X_1=\frac{x_0-MR}{M^2-1},\qquad X_2=\frac{x_0+MR}{M^2-1} \tag{10}$$

$$\begin{aligned}d_1&=\sqrt{X_1^2+r^2}=MX_1+R=M^{-1}(x_0+X_1)\\ d_2&=\sqrt{X_2^2+r^2}=MX_2-R=M^{-1}(x_0+X_2)\end{aligned} \tag{11}$$

are defined, then the kernel functions in Eq. (3) can be summarized as shown in Table 1, where the kernel function vanishes in the upstream region of the Mach cone for the supersonic flow.

In the preceding expression, the function $B_\nu(\nu=1,2)$ has been defined as

$$B_\nu(p,r,X)=\int_{-\infty}^{X}e^{pu}(u^2+r^2)^{-\nu-\frac{1}{2}}\,\mathrm{d}u \tag{12}$$

This function is a type of extended cylindrical function belonging to the special functions. The complex value of the function can be calculated by using an infinite series as follows[13]:

$$\begin{aligned}B_\nu(p,r,X)&=\sum_{m=0}^{\infty}U_m^{(\nu)}(p,r,X)\\ &+\frac{p^{2\nu}}{(2\nu-1)!!\,2^\nu}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(n+\nu)!}\left(\frac{pr}{2}\right)^{2n}\\ &\times\left(\sum_{m=1}^{n}\frac{1}{m}+\sum_{m=n+1}^{n+\nu}\frac{1}{2m}-\gamma^*-\ell n\frac{p}{2}\right)\end{aligned} \tag{13}$$

where γ^* denotes the Euler constant as $\gamma^*=0.5772156649\ldots$.

The terms $U_m^{(\nu)}$ of the summation in Eq. (13) can be calculated through a recursive formula:

$$U_m^{(\nu)}=\frac{p(pX)^{m-1}}{(m-2\nu)m!(X^2+r^2)^{\nu-\frac{1}{2}}}-\frac{(pr)^2}{(m-2\nu)\,m}U_{m-2}^{(\nu)}\qquad(m\neq 2\nu) \tag{14}$$

The initial terms necessary to start the preceding formula are given by

$$U_1^{(\nu)}=\frac{-p}{(2\nu-1)(X^2+r^2)^{\nu-\frac{1}{2}}} \tag{15}$$

$$\begin{aligned}U_{2n}^{(\nu)}&=\frac{(\nu-n-1)!\,p^{2n}}{(2n)!\,r^{2(\nu-n)}}\\ &\times\left[\sum_{m=1}^{\nu-n-1}\frac{(-1)^m}{(2m+2n+1)m!\,(\nu-m-n-1)!}\right.\\ &\times\left.\left(\frac{X}{\sqrt{X^2+r^2}}\right)^{2m+2n+1}+\frac{(2n-1)!!\,2^{\nu-1}}{2^n(2\nu-1)!!}\right]\qquad(n<\nu)\end{aligned} \tag{16}$$

and

$$U_{2\nu}^{(\nu)}=-\frac{p^{2\nu}}{(2\nu)!}\left[\sum_{m=1}^{\nu}\frac{X^{2m-1}}{(2m-1)(X^2+r^2)^{m-\frac{1}{2}}}+\ell n\left(\sqrt{X^2+r^2}-X\right)\right] \tag{17}$$

The following definitions of the double factorial have been used:

$$(2n-1)!!=(2n-1)(2n-3)(2n-5)\ldots 1 \tag{18}$$

and

$$(-1-2n)!!=\frac{(-1)^n}{(2n-1)!!} \tag{19}$$

Reduction to Steady-State Flow

The equations for steady-state flow can be reduced from the equations for unsteady flow by utilizing the final-value theorem of the Laplace transform:

$$\lim_{t\to\infty}w_n(t)=\lim_{p\to 0}pW_n(p)=\lim_{p\to 0}\frac{p}{8\pi}\iint\frac{\ell(\xi,\sigma)}{p}K(p,M)\,\mathrm{d}\xi\,\mathrm{d}\sigma \tag{20}$$

where the symbols w_n and ℓ denote the variable in the time domain and the amplitude of the step lift function, respectively. Therefore, the steady-state kernel can be obtained by putting $p=0$ in the unsteady flow as shown in Table 2.

Continuity at Sonic Flow ($M=1$)

From the forms of the kernel functions shown in Table 2, it is difficult to tell whether they are continuous at $M=1$. Some simple manipulations yield a clearer perspective as shown in Table 3, where the kernel functions vanish in the region of $x_0<0$ for the sonic case.

The continuity of each kernel function listed in Table 3 can be easily verified if we consider the limit

$$\lim_{M\to 1}R=|x_0| \tag{21}$$

DPM at Sonic Flow

The kernel function becomes simply cylindrical for sonic flow as shown in Table 3. If we take a technique similar to the supersonic DPM, the normal wash due to a point doublet can be averaged by an integration over the element area Δ as shown in Fig. 2:

$$w_n=\frac{1}{8\pi\Delta}\iint\left[(\cos\gamma)\frac{2}{r^2}-z'(y'\sin\gamma+z'\cos\gamma)\frac{4}{r^4}\right]\mathrm{d}x_0\,\mathrm{d}y' \tag{22}$$

Table 2 Steady kernel functions

Kernel functions	$M < 1$	$M > 1$
K_1	$\dfrac{M}{Rd} + \dfrac{1}{d(d-X)}$	$\dfrac{M}{Rd_1} + \dfrac{M}{Rd_2} - \dfrac{1}{r^2}\left(\dfrac{X_1}{d_1} - \dfrac{X_2}{d_2}\right)$
K_2	$\dfrac{M(3Rd - M^2r^2)}{R^3d^3} + \dfrac{2d - X}{d^3(d-X)^2}$	$\dfrac{M\left(3Rd_1 - M^2r^2\right)}{R^3d_1^3} - \dfrac{M\left(3Rd_2 + M^2r^2\right)}{R^3d_2^3} - \dfrac{3}{r^4}\left(\dfrac{X_1}{d_1} - \dfrac{X_2}{d_2}\right) + \dfrac{1}{r^4}\left(\dfrac{X_1^3}{d_1^3} - \dfrac{X_2^3}{d_2^3}\right)$

Table 3 Continuity at $M = 1$

Kernel functions	$M < 1$	$M = 1$	$M > 1$
K_1	$\dfrac{1 - M^2}{R(R - x_0)} = \dfrac{R + x_0}{Rr^2}$	$\dfrac{2}{r^2}$	$\dfrac{2x_0}{Rr^2}$
K_2	$\dfrac{2R^3 + 3x_0R^2 - x_0^3}{R^3r^4}$	$\dfrac{4}{r^4}$	$\dfrac{2x_0\left(3R^2 - x_0^2\right)}{R^3r^4}$

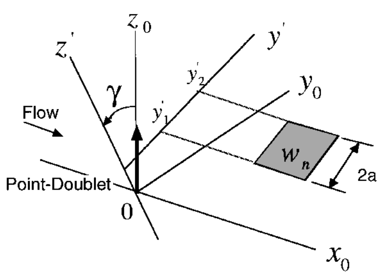

Fig. 2 Integration of normal wash.

The right-hand side of Eq. (22) can be integrated analytically by utilizing the following primitive functions:

$$\int \frac{1}{r^2}\,\mathrm{d}y = \frac{1}{|z|}\tan^{-1}\frac{y}{|z|}, \qquad \int \frac{y}{r^4}\,\mathrm{d}y = \frac{-1}{2(y^2 + z^2)}$$
$$\int \frac{1}{r^4}\,\mathrm{d}y = \frac{y}{2z^2(y^2 + z^2)} + \frac{1}{2z^2|z|}\tan^{-1}\frac{y}{|z|} \qquad (23)$$

The final form of the normal wash becomes

$$w_n = \frac{1}{8\pi(2a)}\left(-\frac{2y'}{r^2}\cos\gamma + \frac{2z'}{r^2}\sin\gamma\right)\Bigg|_{y'=y_1'}^{y'=y_2'} \qquad (24)$$

By following a procedure similar to the supersonic DPM, lift distributions in sonic flow will be obtained with Eq. (24).

Numerical Examples

To examine the continuity of the numerical solutions by the DPM, lift distributions were calculated on rectangular wings with various aspect ratios. The results for the lift curve slopes vs Mach number are shown in Fig. 3, including the well-known analytical solution of the two-dimensional flow depicted with dashed lines. The slender wing theory,[14] which is in a sense derived for sonic flow and does not show the effect of Mach number, may provide relevant references near the unit Mach number. The theory gives

$$C_{L_\alpha} = (\pi/2)AR \qquad (25)$$

where AR denotes the aspect ratio. The corresponding values of Eq. (25) for three wings are plotted at $M = 1$ with solid symbols. The results of the sonic DPM for the rectangular wings agree well with those of the slender wing formula, and the dependency on Mach number decreases as the aspect ratio becomes low. Because the continuity of the numerical solutions at sonic flow is not yet clear with these results, further calculations have been carried out in the proximity of the unit Mach number. The detailed results are shown in Fig. 4. It can be said from the figure that the

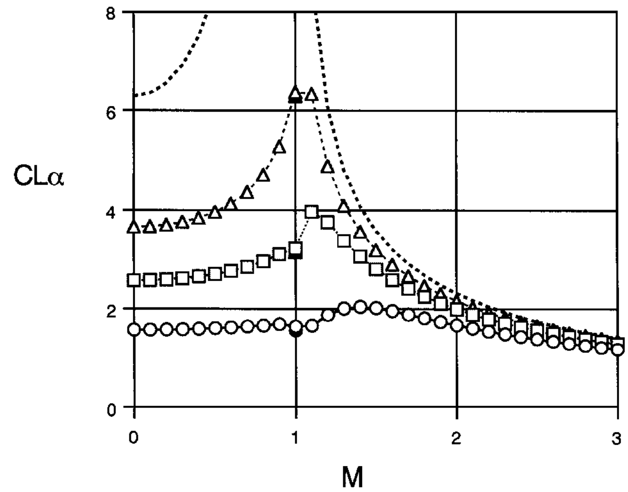

Fig. 3 Lift coefficients of rectangular wings: ○, $AR = 1$; □, $AR = 2$; △, $AR = 4$; - - - -, two dimensional exact; and solid symbols, Eq. (25).

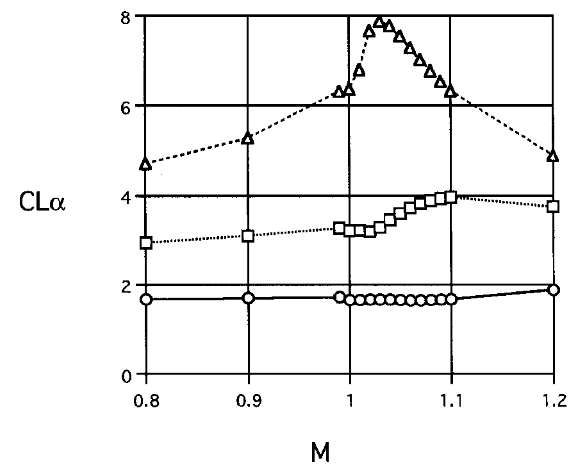

Fig. 4 Detail around $M = 1$: ○, $AR = 1$; □, $AR = 2$; and △, $AR = 4$.

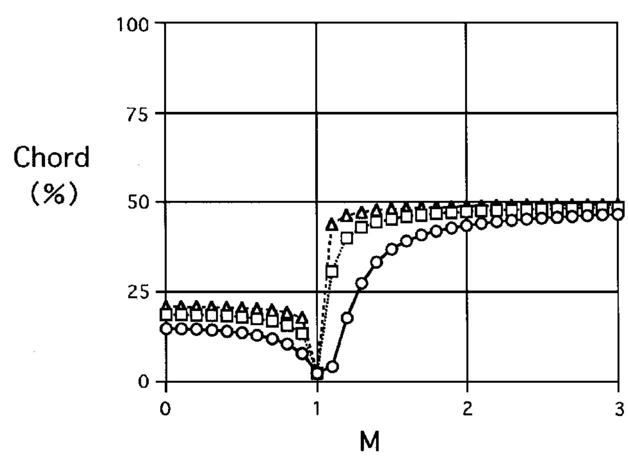

Fig. 5 Location of aerodynamic center: ○, $AR = 1$; □, $AR = 2$; and △, $AR = 4$.

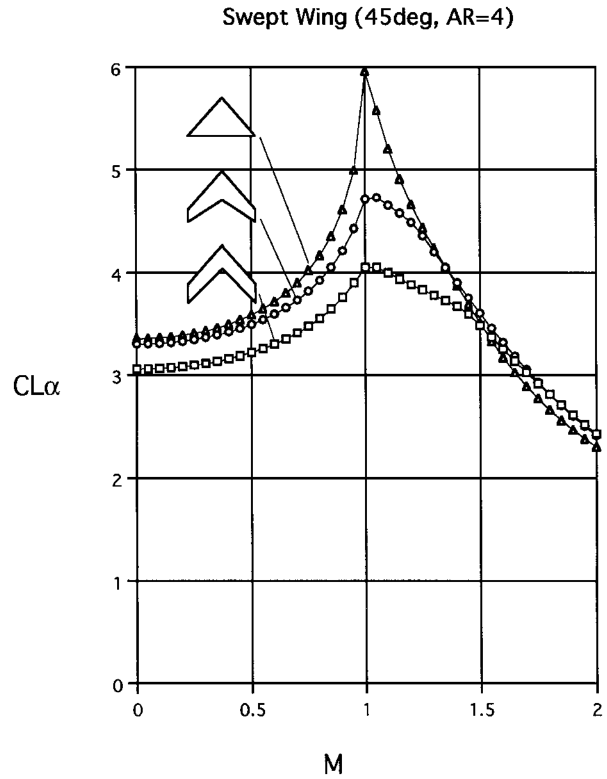

Fig. 6 Lift coefficients of swept wings: □, *TR* = 1; ○, *TR* = 0.5; and △, *TR* = 0.

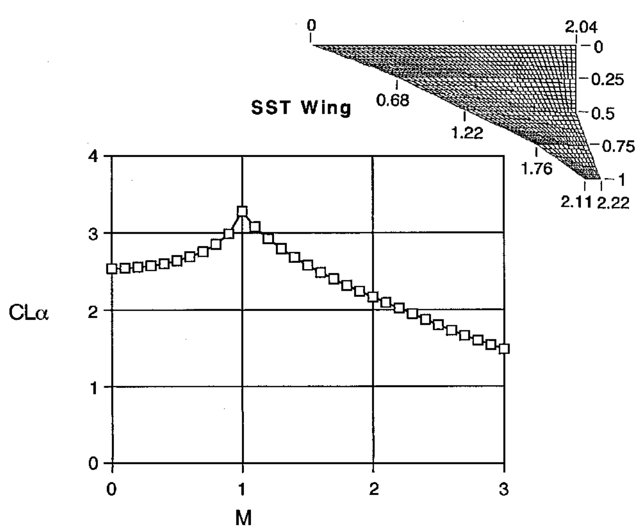

Fig. 7 Lift coefficient of supersonic transport wing: □, DPM.

numerical values obtained by the DPM are quite continuous at $M = 1$. Three different kernel functions have been used for the solutions of subsonic, sonic, and supersonic flow with $M = 0.99$, 1, and 1.01, respectively. Square panels with 20 chordwise elements are used in these calculations. The change of aerodynamic center is also shown in Fig. 5 for the same rectangular wings. It is well known that the aerodynamic center is located at the quarter-chord point in subsonic flow and at the midchord point in supersonic flow for a two-dimensional airfoil that corresponds to an infinite aspect ratio wing. Figure 5 shows how the rectangular wing approaches the two-dimensional airfoil as the aspect ratio increases. The lift distribution of wings of all aspect ratios is concentrated at the leading edge when the flow becomes sonic. The results in the supersonic region agree well with those of the analytical solution.[15]

The dependency of lift coefficients on Mach numbers for several swept wings is shown in Fig. 6. The wings have the same aspect ratio 4 and a sweep-back angle of 45 deg of the leading edge with different taper ratios (*TR*). These are the same wings as those studied in Ref. 16 but with some slight difference in the results.

Finally, a typical result for a supersonic transport wing is depicted in Fig. 7 along with the planform and elements used in the calculation. It can be seen that a smooth lift curve slope has been obtained for all Mach numbers.

Conclusion

The pressure kernel functions of the lifting surface theory for noncoplanar wings were summarized for all Mach numbers, and their continuity at the sonic Mach number was clarified. Numerical examples using these kernel functions and the DPM also show continuous results for the wings in steady flow. Thus one can also verify the continuity for unsteady flow because it includes no singular parameter in the expansion series depending on the reduced frequency.

References

[1]Küssner, H. G., "General Airfoil Theory," NACA TM979, June 1941 (from *Luftfahrtforschung*, Vol. 17, No. 11/12, 1940, pp. 370–378).

[2]Prandtl, L., "Theorie des Flugzeugtragflugels im zusammenduckbaren Medium," *Luftfahrtforschung*, Vol. 13, No. 10, 1936, pp. 313–319.

[3]Watkins, C. E., Runyan, H. L., and Woolston, D. S., "On the Kernel Function of the Integral Equation Relating the Lift and Downwash Distributions of Oscillating Finite Wings in Subsonic Flow," NACA Rept. 1234, Jan. 1955.

[4]Multhopp, H., "Methods for Calculating the Lift Distribution of Wings (Subsonic Lifting-Surface Theory)," Aeronautical Research Council, TR R&M 2884, London, Jan. 1950.

[5]Albano, E., and Rodden, W., "A Doublet Lattice Method for Calculating Lift Distributions on Oscillating Surfaces in Subsonic Flow," *AIAA Journal*, Vol. 7, No. 2, 1969, pp. 279–285.

[6]Cunningham, A. M., Jr., "Oscillatory Supersonic Kernel Function Method for Isolated Wings," *Journal of Aircraft*, Vol. 11, No. 10, 1974, pp. 609–615.

[7]Nissim, E., and Lottati, I., "Supersonic Three-Dimensional Oscillatory Piecewise Continuous Kernel Function Method," *Journal of Aircraft*, Vol. 20, No. 8, 1983, pp. 674–681.

[8]Landahl, M. T., "Kernel Function for Nonplanar Oscillating Surfaces in a Subsonic Flow," *AIAA Journal*, Vol. 5, No. 5, 1967, pp. 1045, 1046.

[9]Harder, R. L., and Rodden, W. P., "Kernel Function for Nonplanar Oscillating Surfaces in Supersonic Flow," *Journal of Aircraft*, Vol. 8, No. 8, 1971, pp. 667–679.

[10]Ueda, T., and Dowell, E. H., "A New Solution Method for Lifting Surface in Subsonic Flow," *AIAA Journal*, Vol. 20, No. 3, 1982, pp. 348–355.

[11]Ueda, T., and Dowell, E. H., "Doublet-Point Method for Supersonic Unsteady Lifting Surfaces," *AIAA Journal*, Vol. 22, No. 2, 1984, pp. 179–186.

[12]Bisplinghoff, R. L., Ashley, H., and Halfman, R. L., *Aeroelasticity*, Addison-Wesley, Reading, MA, 1955, p. 297.

[13]Ueda, T., "Unsteady Aerodynamic Calculations for General Configurations by the Doublet-Point Method," National Aerospace Lab., TR-1101T, Tokyo, Japan, April 1991.

[14]Jones, R. T., and Cohen, D., *High Speed Wing Theory*, Princeton Univ. Press, Princeton, NJ, 1960, p. 215.

[15]Jones, R. T., and Cohen, D., *High Speed Wing Theory*, Princeton Univ. Press, Princeton, NJ, 1960, p. 194.

[16]Woodward, F. A., "Analysis and Design of Wing-Body Configurations at Subsonic and Supersonic Speeds," *Journal of Aircraft*, Vol. 5, No. 6, 1968, pp. 528–534.

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